

**Problem 5094.** Let  $a, b, c$  be real positive numbers such that  $a+b+c+2 = abc$ . Prove that

$$2(a^2 + b^2 + c^2) + 2(a + b + c) \geq (a + b + c)^2$$

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We will use the "magical" substitution given in "Problems from the book" of Titu Andreescu e Gabriel Dospinescu, which is explained in the following

LEMMA. If  $a, b, c$  are positive real numbers such that  $a + b + c + 2 = abc$ , then there exists three real numbers  $x, y, z > 0$  such that

$$a = \frac{y+z}{x}, \quad b = \frac{z+x}{y}, \quad c = \frac{x+y}{z} \quad (*)$$

*Proof.* By means of a simple computation the condition  $a + b + c + 2 = abc$  can be written in the following equivalent form

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1$$

Now if we take

$$x = \frac{1}{1+a}, \quad y = \frac{1}{1+b}, \quad z = \frac{1}{1+c}$$

then  $x + y + z = 1$  and  $a = \frac{1-x}{x} = \frac{y+z}{x}$ . Of course, in the same way we find  $b = \frac{z+x}{y}, c = \frac{x+y}{z}$ . ■

By using the substitution (\*), after some boring calculations, the given inequality rewrites as

$$\frac{z^4(x-y)^2 + x^4(y-z)^2 + y^4(x-z)^2 + 2(x^3y^3 + x^3z^3 + y^3z^3 - 3x^2y^2z^2)}{x^2y^2z^2} \geq 0$$

which is true since

$$x^3y^3 + x^3z^3 + y^3z^3 \geq 3x^2y^2z^2$$

in virtue of AM-GM inequality. □